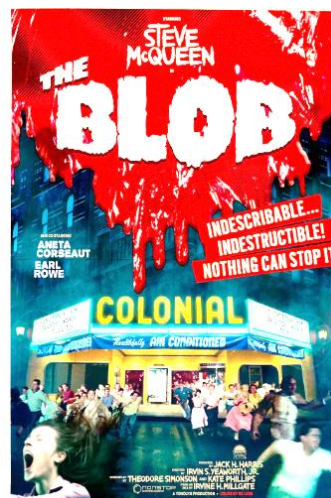

EXPONENTIAL FUNCTIONS

What worried Steve McQueen was not that The Blob was growing by a constant amount every hour, but rather that it was **doubling** in size every hour. If our hero had known his math, he could have warned the town that The Blob was not growing linearly, but *exponentially*.



□ EXPONENTIAL FUNCTIONS

EXAMPLE 1: Analyze The Blob function.

Solution: To give you an idea of the concept of an exponential function, let's look at two scenarios regarding the rate of growth of The Blob. An example of a linear rate of growth might be the formula $B = 4t$, where t is the time in hours and B is the size of The Blob. An exponential formula could be $B = 2^t$. If we construct a table, showing the time and the amount of The Blob for each formula, we can see the true effect of exponential growth.

t	1	2	3	4	5	6	7	8	9	10	11
$4t$	4	8	12	16	20	24	28	32	36	40	44
2^t	2	4	8	16	32	64	128	256	512	1,024	2,048

For the first three hours, there's more blob in the linear formula than in the exponential formula. At $t = 4$, the blob amounts are equal. But after that, it's not even a contest — the exponential

formula shows that The Blob will probably eat the town, the state, and eventually the entire Earth!

EXAMPLE 2: Find some ordered pairs for the exponential function $f(x) = 4^x$.

Solution:

$$\begin{aligned} f(1) &= 4^1 = 4 && \Rightarrow (1, 4) \\ f(-2) &= 4^{-2} = \frac{1}{4^2} = \frac{1}{16} && \Rightarrow (-2, \frac{1}{16}) \\ f(-1) &= 4^{-1} = \frac{1}{4} && \Rightarrow (-1, \frac{1}{4}) \\ f(0) &= 4^0 = 1 && \Rightarrow (0, 1) \\ f(\frac{1}{2}) &= 4^{1/2} = \sqrt{4} = 2 && \Rightarrow (\frac{1}{2}, 2) \\ f(2) &= 4^2 = 16 && \Rightarrow (2, 16) \end{aligned}$$

We now try to determine exactly what the formula for an exponential function looks like, and how it differs from that of a polynomial. Look at the exponential functions given in the previous examples:

$$B = 2^t \qquad f(x) = 4^x$$

Notice that in each function, the base is a constant (the 2 and the 4) and the exponent is a variable (the t and the x). Thus, an **exponential function** is a function of the form

$f(x) = b^x$

constant

↑

variable

where b is some appropriate real number (a constant)

This is in sharp contrast to the notion of a *polynomial*, which is the other way around. Thus,

$y = 10^x$ is an *exponential* function,

$y = x^{10}$ is a *polynomial* function, and

$y = x^x$ is neither an exponential nor a polynomial function.

The question of which real numbers ***b*** serve nicely as the base of an exponential function will be discussed later.

Homework

1. a. Fill in the following chart, similar to Example 1:

t	1	2	3	4	5	6	7	8	9	10
$9t$										
3^t										

- b. Is the $9t$ row of the chart a linear or exponential function?
- c. Is the 3^t row of the chart a linear or exponential function?
- d. For how many hours does the linear growth produce more blob than the exponential growth?
- e. At what hour do both growths give the same amount of blob?
- f. At 10 hours, what is the ratio of the exponential amount of blob to the linear amount of blob?
2. Let's find some more ordered pairs in the function $f(x) = 4^x$ from Example 3.
- a. $(3, \underline{\hspace{1cm}})$ b. $(-3, \underline{\hspace{1cm}})$ c. $(-\frac{1}{2}, \underline{\hspace{1cm}})$ d. $(\frac{3}{2}, \underline{\hspace{1cm}})$
3. Take a guess what the domain of $f(x) = 4^x$ is.

□ GRAPHING EXPONENTIAL FUNCTIONS

Let's use the function $y = 4^x$ discussed in the previous section to make our first exponential graph.

EXAMPLE 3: **Graph:** $y = 4^x$

Solution: Here are some ordered pairs for this function that we found in Example 2 and Homework #2:

$$\left(-1, \frac{1}{4}\right) \quad \left(-\frac{1}{2}, \frac{1}{2}\right) \quad (0, 1) \quad \left(\frac{1}{2}, 2\right) \quad (1, 4) \quad (2, 16)$$

Notice that the point $(0, 1)$ is the **y-intercept**.

Next we analyze a pair of **limits**. First we'll let $x \rightarrow \infty$. As it does, the functional value 4^x approaches ∞ much faster than x does. For example, as x takes the values 6, 8, 10, the y -values go 4096, 65536, 1048576. Wow! The functional values are growing like crazy. The following limit should now be clear:

$$\text{As } x \rightarrow \infty, y \rightarrow \infty \quad [\text{an amazingly fast increase}]$$

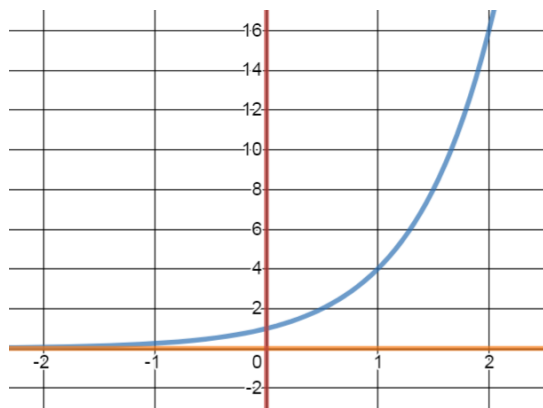
Second, we analyze what the y -values do when $x \rightarrow -\infty$. Consider the three ordered pairs (confirm with your calculator):

$$(-5, 0.000977) \quad (-8, 0.000015) \quad (-10, 0.000000954)$$

It appears that as x grows smaller (towards $-\infty$), the y -values are positive numbers shrinking toward zero. That is,

$$\text{As } x \rightarrow -\infty, y \rightarrow 0$$

The ordered pairs we listed and the limits we calculated lead us to the following graph:



The **domain** is $\mathbb{R}$. Also, there is no vertical **asymptote**, but the line $y = 0$ (the x -axis) is a **horizontal asymptote**.

Last, it appears that there is no **x -intercept** on our graph. Even more importantly, this confirms that

The equation $4^x = 0$ has no solution.

EXAMPLE 4: **Graph:** $y = \left(\frac{1}{2}\right)^x$

Solution: Let's get right to some ordered pairs.

$$\text{If } x = -3, \text{ then } y = \left(\frac{1}{2}\right)^{-3} = \frac{1}{\left(\frac{1}{2}\right)^3} = \frac{1}{\frac{1}{8}} = 8, \text{ which gives us}$$

the ordered pair $(-3, 8)$. It's now your job to verify each of the following ordered pairs in our function:

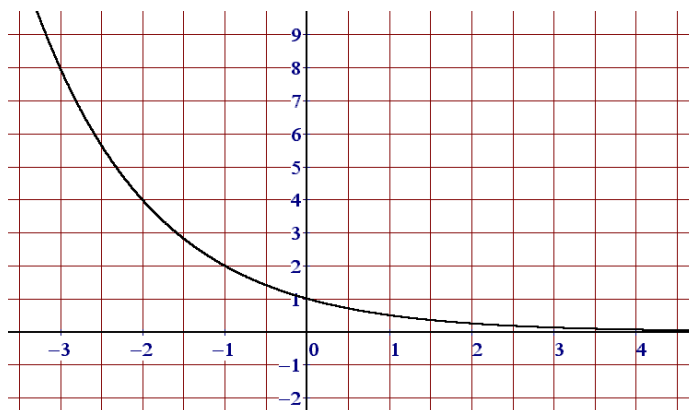
$$\begin{array}{ccccc} (0, 1) & (1, \frac{1}{2}) & (2, \frac{1}{4}) & (3, \frac{1}{8}) & (4, \frac{1}{16}) \\ (-1, 2) & (-2, 4) & (-3, 8) & (-4, 16) & \end{array}$$

The y -intercept is $(0, 1)$. There is no x -intercept, since the equation $\left(\frac{1}{2}\right)^x = 0$ has no solution. The ordered pairs listed above give credence to the following limits:

$$\text{As } x \rightarrow \infty, y \rightarrow 0 \quad \text{and} \quad \text{As } x \rightarrow -\infty, y \rightarrow \infty$$

The ordered pairs and the limits lead us to the following graph:

We can see that the domain of this function is $\mathbb{R}$. Notice also that we have a horizontal asymptote at $y = 0$, but there is no vertical asymptote.



EXAMPLE 5: **Graph:** $f(x) = 2^{-x}$

Solution: Two observations and we'll be done in a jiffy. First, we know that $f(x)$ can be written simply as y . Second, check out the following calculation:

$$2^{-x} = \frac{1}{2^x} = \frac{1^x}{2^x} = \left(\frac{1}{2}\right)^x$$

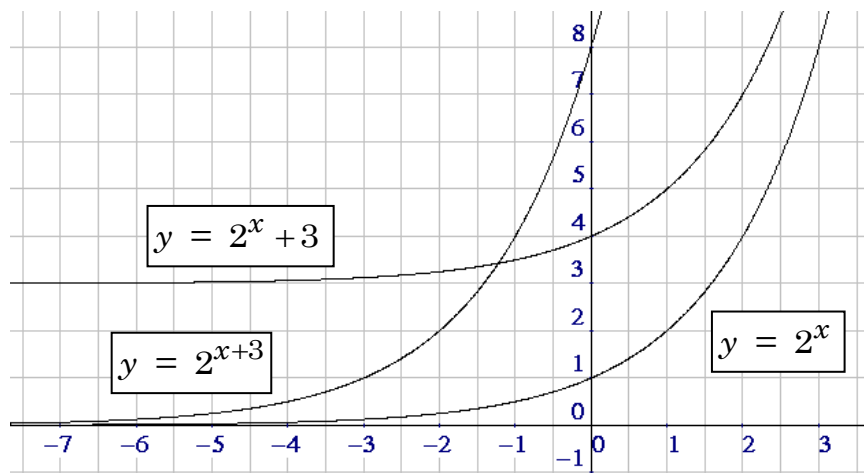
In other words, the original formula can be written $y = \left(\frac{1}{2}\right)^x$,

which we just finished graphing. So the solution to this problem is identical to that of the previous example.

EXAMPLE 6: **Graph:** $y = 2^x$ and $y = 2^x + 3$ and $y = 2^{x+3}$.

Solution: Let's make one table showing x and all the y -values at once and a single grid containing all the graphs at once.

x	2^x	$2^x + 3$	2^{x+3}
-6	1/64	3 1/64	1/8
-5	1/32	3 1/32	1/4
-4	1/16	3 1/16	1/2
-3	1/8	3 1/8	1
-2	1/4	3 1/4	2
-1	1/2	3 1/2	4
0	1	4	8
1	2	5	16
2	4	7	32
3	8	11	64



You should note that the graph of $2^x + 3$ is just the graph of 2^x but shifted 3 units up. Also, we see that the graph of 2^{x+3} is the result of taking the graph of 2^x and shifting it 3 units to the left.

Homework

- | | |
|---------------------------|---|
| 4. Graph: $f(x) = 3^x$ | 5. Graph: $y = 3^x - 2$ |
| 6. Graph: $y = 3^{x+2}$ | 7. Graph: $g(x) = \left(\frac{1}{3}\right)^x$ |
| 8. Graph: $h(x) = 3^{-x}$ | |

❑ THE LEGAL BASES OF AN EXPONENTIAL FUNCTION

In the previous section we graphed exponential functions with bases 4, $\frac{1}{2}$, 2, 3, and $\frac{1}{3}$. Now it's time to figure out exactly which bases we'll allow in the exponential function

$$f(x) = b^x$$

Whatever values of ***b*** we allow to be the base of an exponential function, we'd like the domain of the function (the legal *x*-values) to be

$\mathbb{R}$, the set of real numbers. And we don't want the exponential function to degenerate into some simple function that doesn't possess the "exponential" properties we've seen up until now.

$b < 0$ What about negative bases? Consider $f(x) = (-4)^x$. If we choose $x = \frac{1}{2}$, the functional value is $(-4)^{1/2} = \sqrt{-4}$, not a real number. We thus disallow any base b that is negative.

$b = 0$ Now consider $f(x) = 0^x$. But 0^x is fraught with problems. For example, if $x = 0$, we get 0^0 . Can we assign a value to 0^0 ? On the one hand, 0 to any power should be 0. On the other hand, anything to the 0 power is supposed to be 1. So 0^0 is meaningless (but can be solved in Calculus). Even worse, consider 0^{-2} . Since a negative exponent indicates reciprocal, we get $\frac{1}{0^2} = \frac{1}{0}$, which is undefined. All in all, a base of 0 really stinks.

$0 < b < 1$ These bases are just fine. We used bases of $\frac{1}{2}$ and $\frac{1}{3}$ in the previous section. Even the number $\frac{1}{\pi}$ would be a legal base, although I've never seen it used.

$b = 1$ This gives us the function $f(x) = 1^x$, which is the function $f(x) = 1$, a constant function (the horizontal line $y = 1$). Exponential functions aren't supposed to be flat, so b shouldn't be allowed to be 1.

$b > 1$ Any base bigger than 1 is appropriate. In fact, in computer science a base of **2** is very popular. In basic science, the best base is **10** (for things like acids, earthquakes, and the volume of sound). And in calculus and the more advanced sciences, we use a strange number you may have seen: "**e**".

Homework

9. Describe precisely the legal bases for an exponential function.
10. Explain why -9 is not a good base for an exponential function.
11. Which of the following real numbers are legal bases for an exponential function?

-1 -0.01 0 $\frac{2}{3}$ 0.987 1 π 200

Review Problems

12. T/F: $y = x^3$ is an exponential function.
13. Describe the real numbers which can be used as the base of an exponential function.
14. Explain why 1 is not a good base for an exponential function.
15. Give a function which is both an exponential function and a polynomial function.
16. Graph $y = 5^x$.
17. Graph $y = 3^{-x} - 2$ and state its horizontal asymptote.
18. Let $f(x) = 2^x$. Now let g be the graph which results from taking the graph of f and shifting it 7 units to the left and 4 units up. Find a formula for g .
19. T/F: $y = 3^x$ is an exponential function.
20. T/F: In the exponential function $f(x) = b^x$, b can be any positive real number.

21. What is the domain of the function $y = 10^{7x+1} - 10$?
22. Find all the asymptotes of the function $f(x) = 99^x$.
23. Explain why the graphs of $g(x) = \left(\frac{1}{3}\right)^x$ and $h(x) = 3^{-x}$ are the same.
24. How does the graph of $f(x) = 5^{x-2} + 4$ compare with that of $y = 5^x$?
25. T/F: All exponential functions are increasing functions.
26. Explain why 0 is not a good base for an exponential function.
27. True/False:
 - a. $y = x^3$ is an exponential function.
 - b. $y = \pi^x$ is an exponential function.
 - c. The domain of the function $y = 4^x$ is $[0, \infty)$.
 - d. The function $y = x^x$ is neither polynomial nor exponential.
 - e. The function $y = 2^x - 1$ has an x -intercept.
 - f. For the function above, as $x \rightarrow -\infty$, $y \rightarrow 0$.
 - g. Consider the function $g(x) = \left(\frac{1}{3}\right)^x$. As $x \rightarrow \infty$, $y \rightarrow \infty$.
 - h. Compared to the graph of $y = 4^x$, the graph of $y = 4^{x-5}$ is five units lower.
 - i. Any real number $b > 0$ is a legal base for an exponential function.
 - j. Any real number $b \geq 0$, but not equal to 1, is a legal base for an exponential function.
 - k. The number $\pi + \sqrt{2}$ is a legal base for an exponential function.
 - l. All exponential functions are decreasing functions.

Solutions

1. a.

t	1	2	3	4	5	6	7	8	9	10
$9t$	9	18	27	36	45	54	63	72	81	90
3^t	3	9	27	81	243	729	2187	6561	19683	59049

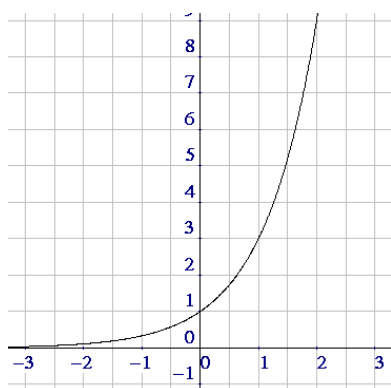
b. linear

c. exponential

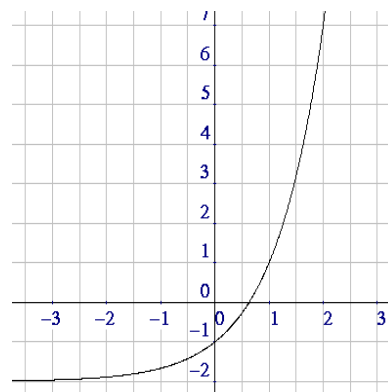
d. first two hours

e. $t = 3$ f. $59,049 / 90 \approx 656$ 2. a. 64 b. $\frac{1}{64}$ c. $\frac{1}{2}$ d. 83. x can be all kinds of numbers, so the domain is probably $\mathbb{R}$.

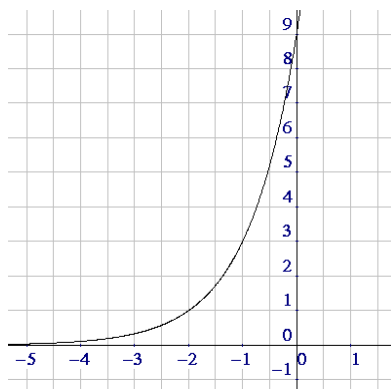
4.



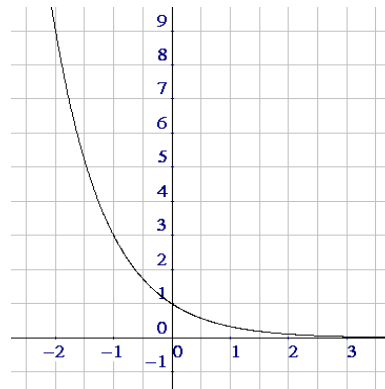
5.



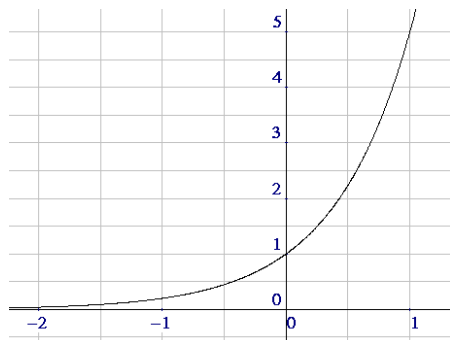
6.



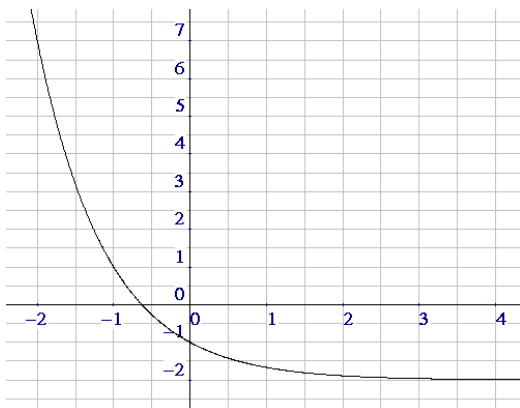
7.



8. Same graph as #14, since $\left(\frac{1}{3}\right)^x = \frac{1^x}{3^x} = \frac{1}{3^x} = 3^{-x}$.
9. The base must be positive but not equal to 1. That is, the function $f(x) = b^x$ is an exponential function if $b > 0$, but $b \neq 1$. In other words, the base b must be in the set: $(0, \infty) - \{1\}$.
10. If we consider the exponential function $y = (-9)^x$, then we could not use $x = \frac{1}{2}$, since $y = (-9)^{\frac{1}{2}} = \sqrt{-9} \notin \mathbb{R}$.
11. $2/3$, 0.987 , π , 200
12. False
13. $\{x \in \mathbb{R} \mid x > 0, x \neq 1\}$ –OR– Any positive real number $\neq 1$
–OR– $(0, 1) \cup (1, \infty)$ –OR– $(0, \infty) - \{1\}$
14. If b were 1, then $f(x) = b^x$ would become $f(x) = 1^x = 1$, which is a simple constant function whose graph is a horizontal line, rather useless to describe exponential growth and decay.
15. Ain't no such animal
- 16.



17.

Horizontal asymptote: $y = -2$

18. $g(x) = 2^{x+7} + 4$

19. T 20. F (any positive real number $\neq 1$)21. $\mathbb{R}$ 22. horiz: $y = 0$; vert: none

23. Because $\left(\frac{1}{3}\right)^x = \frac{1^x}{3^x} = \frac{1}{3^x} = 3^{-x}$.

24. The graph of f is the graph of y shifted 2 units to the right and 4 units up.

25. F

26. Because if $f(x) = 0^x$, then $f(x) = 0$, which is just a horizontal line. Even a better reason: If $f(x) = 0^x$ and we choose $x = -4$, we get an output of 0^{-4} , which is $1/0^4$, which is $1/0$, which is verboten!

27. a. F b. T c. F d. T e. T f. F

g. F h. F i. F j. F k. T l. F

***“The most beautiful experience
we can have is the mysterious.
It is the fundamental emotion
which stands at the cradle
of true art and true science.”***

Albert Einstein